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A Study of a Sequence of Classical Orthogonal Polynomials of Dimension 2

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We construct a sequence of d-dimensional classical orthogonal polynomials ($d \geqslant 2$) that generalize the Gegenbauer polynomials. The case $d=2$ is fully studied. © 1997 Academic Press

1. INTRODUCTION

We give in this paper a partial answer to the problem which consists of the explicit determination of a sequence of polynomials verifying a recurrence relation of order $d+1$ ($d \geqslant 2$).

The problem as it is posed constitutes a generalization of the sequences of classical polynomials, which verify this property (Hermite, Laguerre, Jacobi, and Bessel) when $d=1$ [11, 12].

The relation between the polynomial recurrence relation of order $d+1$ and the notion of orthogonality of dimension d has been established in [9]. The fundamental result in the study of the vectorial Padé approximants of d simultaneous formal sequences is:

"A sequence of polynomials is orthogonal of dimension d iff it verifies a recurrence relation of order $d+1$."

In the paper [1], we have shown the existence of two sequences of "classical" polynomials of dimension 2. These sequences are defined from a Sheffer type generating function.

Part of this work consists of constructing from a generating function a sequence of polynomials verifying a recurrence relation of order $d+1$, where the successive derivatives of order k ($k=1, 2, ...$) verify also a recurrence relation of order $d+1$. This sequence generalizes the Gegenbauer polynomials. On the other hand, our aim is to study the properties of this sequence in the particular case when $d=2$.

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2. THE d-ORTHOGONAL POLYNOMIALS

DEFINITION 2.1 [2, 3, 9, 13, 14]. Let $\Gamma = (\Gamma^1, \Gamma^2, ..., \Gamma^d)^t$ be a d-linear form defined on the vector space of polynomials on C. A sequence $\{P_n\}_{n \geqslant 0}$ is said to be a d-dimensional orthogonal polynomial sequence, or simply d-orthogonal with respect to Γ, if it fulfills

$$\left.\begin{array}{lll} \Gamma^{\sigma}(x^m P_n(x)) = 0, & n \geqslant md + \sigma, & m \geqslant 0 \\ \Gamma^{\sigma}(x^m P_{md+\sigma-1}(x)) \neq 0, & m \geqslant 0, & \end{array}\right\}. \tag{2.1}$$

for each $1 \leqslant \sigma \leqslant d$.

Remark. (a) In this case, the d-dimensional functional Γ is called regular.

(b) If $\{P_n\}_{n \geqslant 0}$ is a d-orthogonal polynomial sequence, then its polynomials are exactly of degree n and can hence be normalized; thus the uniqueness follows.

DEFINITION 2.2 [13]. Let $\{P_n\}_{n \geqslant 0}$ be a sequence of monic polynomials. The sequence of linear forms $\{\mathscr{L}_n\}_{n \geqslant 0}$ defined by

$$\mathscr{L}_n(P_n) = \langle \mathscr{L}_n, P_m \rangle = \delta_{n,m}, \qquad n, m \geqslant 0 \tag{2.2}$$

is called the dual sequence of $\{P_n\}_{n \geqslant 0}$, where $\langle \ , \ \rangle$ denotes the duality bracket between the vector space of polynomials $\mathscr{P}$ and its dual $\mathscr{P}'$.

LEMMA 2.1 [13, 15]. *Let $f \in \mathscr{P}'$ and q be a positive integer. f satisfies*

$$f(P_{q-1}) \neq 0 \qquad \text{and} \qquad f(P_n) = 0, \qquad n \geqslant q \tag{2.3}$$

iff there exist $\lambda_\nu \in C$, for $0 \leqslant \nu \leqslant q-1$, with $\lambda_{q-1} \neq 0$, such that

$$f = \sum_{\nu=0}^{q-1} \lambda_n \mathscr{L}_\nu. \tag{2.4}$$

Remark. From the above lemma we deduce

$$\Gamma^\sigma = \sum_{\nu=0}^{\sigma-1} \lambda_\nu^\sigma \mathscr{L}_\nu, \qquad \text{with} \quad \lambda_{\sigma-1}^\sigma \neq 0 \quad \text{for} \quad 1 \leqslant \sigma \leqslant d, \tag{2.5}$$

or equivalently

$$\mathscr{L}_\nu = \sum_{\sigma=1}^{\nu} \xi_\sigma^\nu \Gamma^\sigma, \qquad \text{with} \quad \xi_\nu^\nu \neq 0 \quad \text{for} \quad 0 \leqslant \nu \leqslant d-1. \tag{2.6}$$

COROLLARY 2.1. *If $\{P_n\}_{n \geqslant 0}$ is a d-orthogonal polynomial sequence with respect to $\mathscr{L} = (\mathscr{L}_0, \mathscr{L}_1, ..., \mathscr{L}_{d-1})^t$, it is therefore d-orthogonal with respect to $\Gamma = (\Gamma^1, \Gamma^2, ..., \Gamma^d)^t$, and reciprocally.*

PROPOSITION 2.1 [9, 13]. *For each sequence $\{P_n\}_{n \geqslant 0}$, the following propositions are equivalent:*

(a) *The sequence $\{P_n\}_{n \geqslant 0}$ is d-orthogonal with respect to $\mathscr{L} = (\mathscr{L}_0, \mathscr{L}_1, ..., \mathscr{L}_{d-1})^t$.*

(b) *The sequence $\{P_n\}_{n \geqslant 0}$ verifies a recurrence relation of order $d+1$,*

$$P_{m+d+1}(x) = (x - \beta_{m+d})\, P_{m+d}(x) - \sum_{\nu=0}^{d-1} \gamma_{m+d-\nu}^{d-1-\nu} P_{m+d-1-\nu}(x), \qquad m \geqslant 0, \tag{2.7}$$

with the initial conditions

$$\begin{aligned} &P_0(x) = 1; \qquad P_1(x) = x - \beta_0; \\ &P_m(x) = (x - \beta_{m-1})\, P_{m-1}(x) - \sum_{\nu=0}^{m-2} \gamma_{m-1-\nu}^{d-1-\nu} P_{m-2-\nu}(x), \qquad 2 \leqslant m \leqslant d \end{aligned} \tag{2.8}$$

and the regularity conditions

$$\gamma_{m+1}^0 \neq 0, \qquad m \geqslant 0.$$

Remark. This result constitutes a generalization of Shohat–Favard's theorem.

DEFINITION 2.3 [5, 6]. The *d*-orthogonal sequence $\{P_n\}_{n \geqslant 0}$ is called "classical" if it satisfies Hahn's property; that is, the sequence $\{DP_n\}_{n \geqslant 0}$ $(D = d/dx)$ is also *d*-orthogonal.

PROPOSITION 2.2. [13]. *If $\{\tilde{\mathscr{L}}_n\}_{n \geqslant 0}$ is the dual sequence of $\{DP_n\}_{n \geqslant 0}$, then*

$$D\tilde{\mathscr{L}}_n = -\mathscr{L}_{n+1}, \qquad n \geqslant 0, \tag{2.9}$$

where

$$\langle D\tilde{\mathscr{L}}_n, p(x) \rangle = -\langle \tilde{\mathscr{L}}_n, p'(x) \rangle, \qquad \forall p \in \mathscr{P}.$$

3. GENERATING FUNCTIONS AND POLYNOMIAL RECURRENCE RELATIONS

DEFINITION 3.1. A function $\Phi(x, t)$ that can be written as a power series in the variable is said to be a generating function for a sequence $\{P_n\}_{n\geqslant 0}$ if it can be represented in the form

$$\Phi(x, t) = \sum_{n\geqslant 0} c_n P_n(x)\, t^n, \qquad c_n \neq 0, \quad n \geqslant 0.$$

LEMMA 3.1. *Let $\{B_n\}_{n\geqslant 0}$ be a sequence of monic polynomials that satisfies a recurrence relation of order $d+1$ $(d\geqslant 2)$, with constant coefficients*

$$\left.\begin{aligned} &B_0(x)=1; \qquad B_j(x)=xB_{j-1}(x)-\sum_{k=1}^{j}\gamma_k B_{j-k}(x), \qquad 1\leqslant j\leqslant d;\\ &B_{n+d+1}(x)=xB_{n+d}(x)-\sum_{k=1}^{d+1}\gamma_k B_{n+d+1-k}(x), \qquad n\geqslant 0, \end{aligned}\right\} \tag{3.1}$$

with $\gamma_{d+1}\neq 0$.

If $G(x, t)$ is a generating function of the sequence $\{B_n\}_{n\geqslant 0}$,

$$G(x, t) = \sum_{n\geqslant 0} B_n(x)\, t^n, \tag{3.2}$$

then

$$G(x, t) = \left(1 - xt + \sum_{k=1}^{d+1} \gamma_k t^k\right)^{-1}. \tag{3.3}$$

Proof. It is sufficient to multiply (3.1) by t^{n+1}, and then to sum over n. ∎

Let us now consider the generating function of the sequence of polynomials denoted by $\{B_n^\alpha\}_{n\geqslant 0}$. It is defined by

$$G_\alpha(x, t) = \left(1 - xt + \sum_{k=1}^{d+1} \gamma_k t^k\right)^{-\alpha} = \sum_{n\geqslant 0} B_n^\alpha(x)\, t^n, \qquad \text{for} \quad n \neq -1, \neq 2, \ldots. \tag{3.4}$$

Remark. The polynomials $B_n^\alpha(x)$ are more general than those of Legendre and Gegenbauer and those studied by Humbert, Pincherle, and Devisme [7].

LEMMA 3.2. *The generating function $G_\alpha(x,t)$ defined by* (3.4) *satisfies the following relations*:

$$\left(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k\right)\frac{\partial G_\alpha}{\partial t}=\alpha\left(x-\sum_{k=1}^{d+1}k\gamma_k t^k\right)G_\alpha(x,t), \tag{3.5}$$

$$\left(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k\right)\frac{\partial G_\alpha}{\partial x}=\alpha t G_\alpha(x,t), \tag{3.6}$$

and

$$t\frac{\partial G_\alpha}{\partial t}=\left(x-\sum_{k=1}^{d+1}k\gamma_k t^{k-1}\right)\frac{\partial G_\alpha}{\partial x}. \tag{3.7}$$

LEMMA 3.3. *The sequence $\{B_n^\alpha\}_{n\geqslant 0}$ satisfies the following recurrence relation of order $d+1$*:

$$\left.\begin{aligned}
&B_0^\alpha(x)=1;\\
&jB_j^\alpha(x)=(j-1+\alpha)\,xB_{j-1}^\alpha(x)-\sum_{k=1}^{j}(j-k+k\alpha)\,\gamma_k B_{j-k}^\alpha(x),\\
&\qquad 1\leqslant j\leqslant d;\\
&(n+1+d)\,B_{n+d+1}^\alpha(x)\\
&\quad=(n+d+\alpha)\,xB_{n+d}^\alpha(x)-\sum_{k=1}^{d+1}(n+1+d+k\alpha-k)\,\gamma_k B_{n+d+1-k}^\alpha(x),\\
&\qquad n\geqslant 0.
\end{aligned}\right\} \tag{3.8}$$

Proof. It is sufficient to replace $\partial G_\alpha/\partial t$ and G_α in (3.5) by their respective values, and then we identify the coefficients of power of t. ∎

LEMMA 3.4. *The sequence $\{B_n^\alpha\}_{n\geqslant 0}$ satisfies the following relations*:

$$\left.\begin{aligned}
&\alpha B_j^\alpha(x)=DB_{j+1}^\alpha(x)-xDB_j^\alpha(x)+\sum_{k=1}^{j}\gamma_k DB_{j+1-k}^\alpha(x),\\
&\qquad 1\leqslant j\leqslant d;\\
&\alpha B_{n+d+1}^\alpha(x)=DB_{n+d+2}^\alpha(x)-xDB_{n+d+1}^\alpha(x)+\sum_{k=1}^{d+1}\gamma_k DB_{n+d+2-k}^\alpha(x),\\
&\qquad n\geqslant 0,
\end{aligned}\right\} \tag{3.9}$$

and

$$\left.\begin{aligned} kB_j^\alpha(x) &= xDB_j^\alpha(x) - \sum_{k=1}^{j} k\gamma_k DB_{j+1-k}^a(x), \\ &\qquad 1 \leqslant j \leqslant d; \\ (n+d+1)\, B_{n+d+1}^\alpha(x) &= xDB_{n+d+1}^\alpha(x) - \sum_{k=1}^{d+1} k\gamma_k DB_{n+d+2-k}^\alpha(x), \\ &\qquad n \geqslant 0. \end{aligned}\right\} \tag{3.10}$$

Proof. It is sufficient to replace $\partial G_\alpha/\partial x$ and G_α in (3.6) by their respective values, and then we identify the coefficients of power of t to obtain (3.9).

Similarly, we obtain (3.10) by replacing $\partial G_\alpha/\partial x$ and $\partial G_\alpha/\partial t$ in (3.7) by their respective values. ∎

COROLLARY 3.1. *Differentiating the relations* (3.9) *and* (3.10) m *times* $(m \leqslant n)$, *and letting* $D^m = d^m/dx^m$, *we obtain the following relations for* $0 \leqslant m \leqslant n$, *with* $n \geqslant 0$:

$$(\alpha+m)\, D^m B_{n+d+1}^\alpha(x) = D^{m+1} B_{n+d+2}^\alpha(x) - xD^{m+1} B_{n+d+1}^\alpha(x) + \sum_{k=1}^{d+1} \gamma_k D^{m+1} B_{n+d+2-k}^\alpha(x), \tag{3.11}$$

$$(n+d+1-m)\, D^m B_{n+d+1}^\alpha(x) = xD^{m+1} B_{n+d+2}^\alpha(x) - \sum_{k=1}^{d+1} k\gamma_k D^{m+1} B_{n+d+2-k}^\alpha(x). \tag{3.12}$$

THEOREM 3.1. *The sequence of derivatives* $\{D^{m+1}B_n^\alpha\}_{n\geqslant 0}$, $(m<n)$ *also satisfies a recurrence relation of order* $d+1$:

$$\begin{aligned} (n+d+1-m)\, D^{m+1} B_{n+d+2}^\alpha(x) &= (n+d+1+\alpha)\, xD^{m+1} B_{n+d+1}^\alpha(x) \\ &\quad - \sum_{k=1}^{d+1} [n+d+1+k\alpha-(k-1)m]\, \gamma_k D^{m+1} B_{n+d+2-k}^\alpha(x), \\ &\qquad 0 \leqslant m \leqslant n; \quad n \geqslant 0. \end{aligned} \tag{3.13}$$

Proof. We cancel $D^m B_{n+d+1}(x)$ by taking a linear combination of (3.11) and (3.12). ∎

Remark. It follows that the sequence $\{B_n^\alpha\}_{n\geqslant 0}$ $(\alpha \neq -1, \neq -2, ...)$ is a sequence of d-dimensional classical orthogonal polynomials.

4. PROPERTIES OF $\{B_n^\alpha\}_{n\geqslant 0}$

Remarks. (a) From the generating function (3.4) and Cauchy's integral formula, $\{B_n^\alpha\}_{n\geqslant 0}$ can be written in the form

$$B_n^\alpha(x)=\frac{1}{2\pi i}\oint\frac{dt}{t^{n+1}(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k)^\alpha}$$
$$=\frac{1}{2\pi i\gamma_{d+1}^\alpha}\oint\frac{dt}{t^{n+1}\prod_{k=1}^{d+1}[t-\tau_k(x)]^\alpha}, \tag{4.1}$$

where $\tau_1(x), \tau_2(x),..., \tau_{d+1}(x)$ are the $(d+1)$ zeros of

$$1-x\tau+\sum_{k=1}^{d+1}\gamma_k\tau^k=0, \tag{4.2}$$

with $|\tau_1(x)|\leqslant|\tau_2(x)|\leqslant...\leqslant|\tau_{d+1}(x)|$.

(b) We can see that $B_n^\alpha(x)$ behaves like the nth power of $1/\tau_1(x)$.

LEMMA 4.1. *The recurrence relation* (3.8) *can be written in the form*

$$x\mathbf{b}=\mathbf{M}\mathbf{b}, \tag{4.3}$$

where

$$\mathbf{b}=\begin{bmatrix} B_0^\alpha \\ B_1^\alpha(x) \\ \vdots \end{bmatrix}$$

and

$$\mathbf{M}=\begin{bmatrix} \gamma_1 & \frac{1}{\alpha} & 0 & 0 & \cdots \\ \frac{2\alpha}{\alpha+1}\gamma_2 & \gamma_1 & \frac{2}{\alpha+1} & 0 & \cdots \\ \frac{3\alpha}{\alpha+2}\gamma_3 & \frac{2\alpha+1}{\alpha+2}\gamma_2 & \gamma_1 & \frac{3}{\alpha+2} & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \frac{(d+1)\alpha}{\alpha+d}\gamma_{d+1} & \frac{d\alpha+1}{\alpha+d}\gamma_d & \frac{(d-1)\,\alpha+2}{\alpha+d}\gamma_{d-1} & \cdots & \cdots \\ 0 & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \end{bmatrix}.$$

PROPOSITION 4.1. *The moments of $\mathscr{L}_v$ are given by*

$$\mathscr{L}_v(x^n) = M^n_{0,\,v}, \qquad n \geqslant 0, \tag{4.4}$$

where $M^n_{0,\,v}$ is the element of the first line and the $(v+1)$th *column of* $\mathbf{M}^n$.

Proof. Multiplying the relation (4.3) $(n-1)$ times by $\mathbf{M}$, we obtain

$$x^n \mathbf{b} = \mathbf{M}^n \mathbf{b}.$$

In particular

$$x^n = \sum_{j \geqslant 0} M^n_{0,j} B^\alpha_j(x).$$

Applying now $\mathscr{L}_v$, we get

$$\mathscr{L}_v(x^n) = \sum_{j \geqslant 0} M^n_{0,j} \mathscr{L}_v(B^\alpha_j(x)) = M^n_{0,\,v}. \quad \blacksquare$$

LEMMA 4.2. *The forms $\{\mathscr{L}_v\}_{v \geqslant 0}$ satisfy the relation*

$$\frac{v \mathscr{L}_{v-1}(p)}{\alpha + v - 1} - (\alpha + v)\, \frac{\mathscr{L}_v(xp)}{\alpha + v} + \sum_{k=1}^{d+1} (k\alpha + v)\, \gamma_k \frac{\mathscr{L}_{v+k-1}(p)}{\alpha + v + k - 1} = 0, \qquad \forall p \in \mathscr{P}. \tag{4.5}$$

Proof. We have

$$x^{n+1} \mathbf{b} = \mathbf{M}^{n+1} \mathbf{b}.$$

In particular

$$x^{n+1} = \sum_{j \geqslant 0} M^{n+1}_{0,j} B^\alpha_j(x) = \sum_{j \geqslant 0} M^n_{0,j} x B^\alpha_j(x).$$

Applying now $\mathscr{L}_v$, we get

$$\mathscr{L}_v(x^{n+1}) = M^{n+1}_{0,\,v} = \frac{v}{\alpha + v - 1} M^n_{0,\,v-1} + \sum_{k=1}^{d+1} \frac{k\alpha + v}{\alpha + v + k - 1} \gamma_k M^n_{0,\,v+k-1}.$$

That is,

$$\mathscr{L}_v(x^{n+1}) = \frac{v}{\alpha + v - 1} \mathscr{L}_{v-1}(x^n) + \sum_{k=1}^{d+1} \frac{k\alpha + v}{\alpha + v + k - 1} \gamma_k \mathscr{L}_{v+k-1}(x^n),$$

and by the linearity of $\mathscr{L}_v$, this relation is true for any polynomial p; thus (4.5) follows. $\blacksquare$

LEMMA 4.5. *Let* $\{\tilde{\mathscr{L}}_v\}_{v\geqslant 0}$ *be the dual sequence of* $\{DB_{n+1}\}_{n\geqslant 0}$; *then*

$$\mathscr{L}_{v+1}(p)=\tilde{\mathscr{L}}_v(p')=\frac{\mathscr{L}_v(p')}{\alpha+v}-\sum_{j=2}^{d+1}(j-1)\,\gamma_j\frac{\mathscr{L}_{v+j}(p')}{\alpha+v+j}, \qquad \forall p\in\mathscr{P}. \tag{4.6}$$

Proof. Adding term by term the relations (3.9) and (3.10), we obtain

$$(\alpha+k)\,B_k^\alpha(x)=DB_{k+1}^\alpha(x)-\sum_{j=2}^{\min(k,\,d+1)}(j-1)\,\gamma_j DB_{k+1-j}^\alpha(x), \qquad n\geqslant 0.$$

Applying now $\tilde{\mathscr{L}}_v$, we get

$$\tilde{\mathscr{L}}_v(B_k(x))=\frac{\delta_{k,\,v}}{\alpha+v}-\sum_{j=2}^{\min(k,\,d+1)}(j-1)\,\gamma_j\frac{\delta_{k,\,v+j}}{\alpha+v+j}.$$

Thus

$$\tilde{\mathscr{L}}_{v+1}=\frac{\mathscr{L}_v}{\alpha+v}-\sum_{j=2}^{d+1}(j-1)\,\gamma_j\frac{\mathscr{L}_{v+j}}{\alpha+v+j},$$

and by the linearity of $\mathscr{L}_v$ and $\tilde{\mathscr{L}}_v$ this relation is true for any polynomial p; thus (4.6) follows. ▮

THEOREM 4.1. *The forms* $\mathscr{L}_v/(\alpha+v)$ *have an integral representation in the form*

$$\frac{\mathscr{L}_v(f)}{\alpha+v}=\int_{x_1}^{x_2} w_v(x)\,f(x)\,dx=\int_{x_1}^{x_2}\int_{t_1(x)}^{t_2(x)} t^v K(x,t)\,f(x)\,dt\,dx, \tag{4.7}$$

with

$$K(x,t)=constant\left(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k\right)^{\alpha-1}$$
$$=constant\prod_{k=1}^{d+1}[t-t_k(x)]^{\alpha-1}, \qquad if \quad \alpha>0.$$

That is,

$$w_v(x)=\int_{t_1(x)}^{t_2(x)} t^v\prod_{k=1}^{d+1}[t-t_k(x)]^{\alpha-1}\,dt, \tag{4.8}$$

where $t_1(x)=\tau_1(x)$, $t_2(x)=\tau_2(x)$, ... *and* x_1 *and* x_2 *are two values such that* $\tau_1(x)=\tau_2(x)$.

Proof. From (4.5) we have

$$\int_{x_1}^{x_2}\int_{t_1(x)}^{t_2(x)} t^{\nu}K(x,t)\left[\frac{\nu}{t}-(\alpha+\nu)x+\sum_{k=1}^{d+1}(k\alpha+\nu)\,\gamma_k t^{k-1}\right]dt\,f(x)\,dx=0.$$

This implies that

$$\int_{x_1}^{x_2}\int_{t_1(x)}^{t_2(x)} t^{\nu}K(x,t)\,\frac{(\partial/\partial t)[t^{\nu}(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k)^{\alpha}]}{t^{\nu}(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k)^{\alpha-1}}\,dt\,f(x)\,dx=0.$$

Hence, it is sufficient to take

$$K(x,t)=h(x)\left(1-xt+\sum_{k=1}^{d+1}\gamma_k t^k\right)^{\alpha-1},$$

and to have

$$1-xt+\sum_{k=1}^{d+1}\gamma_k t^k=0,\qquad \text{for}\quad t=t_1(x)\text{ and }t=t_2(x),\quad (\text{if }\alpha>0).$$

Thus the limits of integration are $t_1(x)=\tau_1(x)$ and $t_2(x)=\tau_2(x)$; in addition x_1 and x_2 are two values such that $\tau_1(x)=\tau_2(x)$.

To determine $h(x)$, let $W_{\nu+1}$ be a primitive of $w_{\nu+1}$; then we have

$$\begin{aligned}\mathscr{L}_{\nu+1}(f)&=(\nu+\alpha+1)\int_{x_1}^{x_2} w_{\nu+1}(x)\,f(x)\,dx\\&=-(\nu+\alpha+1)\int_{x_1}^{x_2} W_{\nu+1}(x)\,f'(x)\,dx,\end{aligned}$$

and from (4.6), we get

$$\begin{aligned}&-(\nu+\alpha+1)\,W_{\nu+1}(x)\\&\quad=w_{\nu}(x)-\sum_{j=2}^{d=1}(j-1)\,\gamma_j w_{\nu+j}(x)\\&\quad=h(x)\int_{t_1(x)}^{t_2(x)} t^{\nu}\left(1-xt+\sum_{j=1}^{d+1}\gamma_j t^j\right)^{\alpha-1}\left[1-\sum_{j=2}^{d+1}(j-1)\,\gamma_j t^j\right]dt\\&\quad=-h(x)\int_{t_1(x)}^{t_2(x)} t^{\nu+\alpha+1}\left[t^{-1}\left(1-xt+\sum_{j=1}^{d+1}\gamma_j t^j\right)\right]^{\alpha-1}\\&\qquad\times\frac{\partial}{\partial t}\left[t^{-1}\left(1-xt+\sum_{j=1}^{d+1}\gamma_j t^j\right)\right]dt\\&\quad=\frac{(\nu+\alpha+1)}{\alpha}\,h(x)\int_{t_1(x)}^{t_2(x)} t^{\nu}\left[1-xt+\sum_{j=1}^{d+1}\gamma_j t^j\right]^{\alpha}dt.\end{aligned}$$

Differentiating the last expression with respect to x, we obtain (4.8) with $h(x)=\text{constant}$. ∎

Remark. When $d=1$, we obtain

$$w_0(x)=\text{constant}[t_2(x)-t_1(x)]^{2\alpha-1},$$

which is the density function of Gegenbauer's polynomials.

5. STUDY OF THE CASE $d=2$

Corollary 5.1. *If we put* $\gamma_1=\beta$, $\gamma_2=\gamma$, *and* $\gamma_3=\delta$, *we obtain from relations* (3.4), (3.8), *and* (3.13), *with* $d=2$,

$$\begin{aligned} G_\alpha(x,t) &= [1-(x-\beta)\,t+\gamma t^2+\delta t^3]^{-\alpha} \\ &= \sum_{n\geqslant 0} B_n^\alpha(x)\,t^n, \qquad \textit{for}\quad n\neq -1,\ \neq -2,\ldots \end{aligned} \tag{5.1}$$

$$\left.\begin{aligned} &B_0^\alpha(x)=1; \qquad B_1^\alpha(x)=\alpha(x-\beta); \\ &B_2^\alpha(x)=\frac{\alpha}{2}[(\alpha+1)(x-\beta)^2-2\gamma]; \\ &(n+3)\,B_{n+3}^\alpha(x)=(n+2+\alpha)(x-\beta)\,B_{n+2}^\alpha(x)-(n+1+2\alpha)\,\gamma B_{n+1}^\alpha(x) \\ &\qquad\qquad -(n+3\alpha)\,\delta B_n^\alpha(x), \qquad n\geqslant 0 \end{aligned}\right\} \tag{5.2}$$

and

$$\begin{aligned} (n+3)\,D^{m+1}B_{n+4}^\alpha(x) &= (n+3+\alpha)(x-\beta)\,D^{m+1}B_{n+3}^\alpha(x) \\ &\quad -(n+3+2a+m)\,\gamma D^{m+1}B_{n+2}^\alpha(x) \\ &\quad -(n+3+3\alpha+2m)\,\delta D^{m+1}B_{n+1}^\alpha(x), \\ &\qquad 0\leqslant m\leqslant n;\quad n\geqslant 0. \end{aligned} \tag{5.3}$$

Corollary 5.2. *Let*

$$B_n^\alpha(x)=\frac{[\alpha]_n}{n!}\,\tilde{B}_n^\alpha(x) \qquad \textit{and} \qquad Q_n^\alpha(x)=\frac{D\tilde{B}_{n+1}^\alpha(x)}{n+1},$$

where $[\alpha]_n=\alpha(\alpha+1)\cdots(\alpha+n-1)$, $n\geqslant 1$, *and* $[\alpha]_0=1$.

Then the sequences $\{\tilde{B}_n^\alpha\}_{n\geqslant 0}$ *and* $\{Q_n^\alpha\}_{n\geqslant 0}$ *are two monic sequence. Then satisfy, respectively, the following recurrence relations*:

$$\left.\begin{aligned}
&\tilde{B}_0^\alpha=1; \qquad \tilde{B}_1^\alpha(x)=x-\beta; \qquad \tilde{B}_2^\alpha(x)=(x-\beta)^2-\frac{2\gamma}{\alpha+1};\\
&\tilde{B}_{n+3}^\alpha(x)=(x-\beta)\,\tilde{B}_{n+2}^\alpha(x)-\frac{(n+2)(n+1+2\alpha)}{(n+1+\alpha)(n+2+\alpha)}\,\gamma\tilde{B}_{n+1}^\alpha(x)\\
&\qquad -\frac{(n+1)(n+2)(n+3\alpha)}{(n+\alpha)(n+1+\alpha)(n+2+\alpha)}\,\delta\tilde{B}_n^\alpha(x), \qquad n\geqslant 0
\end{aligned}\right\} \tag{5.4}$$

and

$$Q_{n+3}(x)=(x-\beta)\,Q_{n+2}(x)-\frac{(n+2)(n+3+2\alpha)}{(n+2+\alpha)(n+3+\alpha)}\,\gamma Q_{n+1}(x)$$
$$-\frac{(n+1)(n+2)(n+3+3\alpha)}{(n+1+\alpha)(n+2+\alpha)(n+3+\alpha)}\,\delta Q_n(x), \qquad n\geqslant 0. \tag{5.5}$$

Remarks. (a) The sequence $\{\tilde{B}_n^\alpha\}_{n\geqslant 0}$ corresponds to the case A in [5] (which is not 2-symmetric when $\beta=0$, because it can be concluded that $\gamma_{n+1}^1\neq 0$), and from this, the conclusions concerning this case are not complete.

(b) The relations of Section 3 are between the polynomials of the same index. If we omit this restriction, we can find other sample relations that generalize the classical identities of Gegenbauer's polynomials. Note, for example,

$$D^m B_n^\alpha(x)=(-1)^m\,[\alpha]_m\,B_{n-m}^{\alpha+m}(x); \tag{5.6}$$

$$Q_n^\alpha(x)=\tilde{B}_n^{\alpha+1}(x); \tag{5.7}$$

$$\tilde{B}_n^{\alpha+1}(x)=\frac{n!}{(n+\alpha)!}\,D^\alpha\tilde{B}_{n+\alpha}^\alpha(x), \qquad \text{if} \quad \alpha\in N; \tag{5.8}$$

$$\tilde{B}_{n+3}^\alpha(x)=(x-\beta)\,\tilde{B}_{n+2}^{\alpha+1}(x)-\frac{2(n+2)}{n+2+\alpha}\,\gamma\tilde{B}_{n+1}^{\alpha+1}(x)$$
$$-\frac{3(n+1)(n+2)}{(n+1+\alpha)(n+2+\alpha)}\,\delta\tilde{B}_n^{\alpha+1}(x); \tag{5.9}$$

and

$$\tilde{B}^{\alpha}_{n+3}(x) = \frac{1}{3(n+2+\alpha)} \Big\{ 2(n+2)[(x-\beta)^2-\gamma]\, \tilde{B}^{\alpha+1}_{n+1}(x) - \frac{(n+1)(n+2)}{n+1+\alpha} [\gamma(x-\beta)+2\delta]\, \tilde{B}^{\alpha+1}_{n} + ((n+2+3\alpha)(x-\beta)\, \tilde{B}^{\alpha}_{n+2}(x)) \Big\}, \quad n \geqslant 0. \tag{5.10}$$

PROPOSITION 5.1. *If the equation*

$$Z(x, \tau) = 1-(x-\beta)\,\tau + \gamma\tau^2 + \delta\tau^3 = 0. \tag{5.11}$$

has a double root $\tau_1(x) = \tau_2(x)$, then x must satisfy the equation

$$P(x) = -4\delta(x-\beta)^3 - \gamma^2(x-\beta)^2 + 18\gamma\delta(x-\beta) + 4\gamma^3 + 27\delta^2 = 0. \tag{5.12}$$

Proof. We have

$$\left.\begin{aligned} &\frac{\partial Z}{\partial \tau} = -(x-\beta) + 2\gamma\tau + 3\delta\tau^2 = 0, \\ &Z(x, t) = 1-(x-\beta)\tau + \gamma\tau^2 + \delta\tau^3 = 0. \end{aligned}\right\}$$

Thus $\tau = (9\delta + (x-\beta)\gamma)/2[3\delta(x-\beta)+\gamma^2]$, and if we replace τ by this value, we obtain (5.12). ∎

PROPOSITION 5.2. *If* $\alpha \in N$ $(\alpha > 0)$, *then*

$$\begin{aligned} w_0(x) &= \frac{(-1)^{\alpha}}{\alpha} [t_2(x)-t_1(x)]^{2\alpha-1} \sum_{k=0}^{\alpha-1} \frac{\binom{k}{\alpha-1}}{\binom{\alpha}{2\alpha+k-1}} \times [t_1(x)-t_3(x)]^{\alpha-1-k}\, [t_2(x)-t_1(x)]^{k} \\ &= \frac{(-1)^{\alpha}}{\alpha} [t_2(x)-t_1(x)]^{2\alpha-1} \sum_{j=0}^{\alpha-1} \left\{ \sum_{k=j}^{\alpha-1} (-1)^{j+k} \frac{\binom{k}{\alpha-1}\binom{j}{k}}{\binom{\alpha}{2\alpha+k-1}} \right\} \\ &\quad \times [t_2(x)-t_3(x)]^{j}\, [t_1(x)-t_3(x)]^{\alpha-1-j}, \end{aligned} \tag{5.13}$$

and

$$w_1(x)=(-1)^{\alpha}\,[t_2(x)-t_1(x)]^{2\alpha-1}\sum_{k=0}^{\alpha-1}\frac{\binom{k}{\alpha-1}}{\binom{\alpha-1}{2\alpha+k}}$$
$$\times[t_1(x)-t_3(x)]^{\alpha-1-k}\,[t_2(x)-t_1(x)]^k\,[(\alpha+k)\,t_2(x)+\alpha t_1(x)]. \tag{5.14}$$

Proof. We have

$$\begin{aligned}
w_0(x)&=\int_{t_1(x)}^{t_2(x)}[t-t_1(x)]^{\alpha-1}\,[t-t_2(x)]^{\alpha-1}\,[t-t_3(x)]^{\alpha-1}\,dt\\
&=\int_{t_1(x)}^{t_2(x)}[t-t_1(x)]^{\alpha-1}\,[t-t_2(x)]^{\alpha-1}\sum_{k=0}^{\alpha-1}\binom{k}{\alpha-1}\\
&\quad\times[t-t_1(x)]^k\,[t_1(x)-t_3(x)]^{\alpha-1-k}\,dt\\
&=\sum_{k=0}^{\alpha-1}\binom{k}{\alpha-1}[t_1(x)-t_3(x)]^{\alpha-1-k}\\
&\quad\times\int_{t_1(x)}^{t_2(x)}[t-t_1(x)]^{\alpha-1+k}\,[t-t_2(x)]^{\alpha-1}\,dt\\
&=\frac{(-1)^{\alpha}}{\alpha}[t_2(x)-t_1(x)]^{2\alpha-1}\sum_{k=0}^{\alpha-1}\frac{\binom{k}{\alpha-1}}{\binom{\alpha}{2\alpha+k-1}}\\
&\quad\times[t_1(x)-t_3(x)]^{\alpha-1-k}\,[t_2(x)-t_1(x)]^k.
\end{aligned}$$

The relation (5.14) can be obtained similarly, by writting

$$w_1(x)=\int_{t_1(x)}^{t_2(x)}\{[t-t_1(x)]+t_1(x)\}[t-t_1(x)]^{\alpha-1}\,[t-t_2(x)]^{\alpha-1}$$
$$\times[t-t_3(x)]^{\alpha-1}\,dt. \qquad \blacksquare$$

THEOREM 5.1. *The sequence of polynomials* $\{B_n^{\alpha}\}_{n\geqslant 0}$ *satisfies the following third-order differential equation,*

$$r_{1,n}(x)\,S_3(x)\,Y^{(3)}+b_{3,n}(x)\,Y''+c_{2,n}(x)\,Y'+d_{1,n}(x)\,Y=0, \tag{5.15}$$

with

$$\left.\begin{aligned} &S_3(x)=3P(x),\\ &r_{1,n}(x)=3(3n+3\alpha+1)\,\gamma\delta x+2n\gamma^3+27(n+1+3\alpha)\,\delta^2,\\ &b_{3,n}(x)=\frac{2\alpha+3}{2}\,DS_3(x)\,r_{1,n}(x)-Dr_{1,n}S_3(x),\\ &c_{2,n}(x)=3\{[(n-2-3\alpha)(n+5+3\alpha)+2n(n+3\alpha)]\,\delta x\\ &\qquad\qquad+(n-1)(n+1+2\alpha)\,\gamma^2\}\,r_{1,n}(x)\\ &\qquad\qquad-[6(n-2-3\alpha)\,\delta x^2+(n-3-6\alpha)\,\gamma^2 x-9(n-1)\,\gamma\delta]\,Dr_{1,n},\\ &d_{1,n}(x)=n(n+3\alpha)[3(n+3+3\alpha)\,\delta r_{1,n}(x)-(6\delta x+2\gamma^2)\,Dr_{1,n}], \end{aligned}\right\} \tag{5.16}$$

and the substitution $x\mapsto x+\beta$.

Proof. Differentiating the relation (5.2) with n replaced by $n-2$, we have

$$\text{(R1)}\qquad \begin{aligned}&(n+1)\,DB^{\alpha}_{n+1}-(n+\alpha)\,xDB^{\alpha}_{n}+(n+2\alpha-1)\,\gamma DB^{\alpha}_{n-1}\\ &\qquad+(n+3\alpha-2)\,\delta DB^{\alpha}_{n-2}-(n+\alpha)\,B^{\alpha}_{n}=0.\end{aligned}$$

By eliminating DB^{α}_{n-2}, using the relations (R1) and (3.11), in which we replace m by 0 and change n to $n-3$, we obtain

$$\text{(R2)}\qquad 3DB^{\alpha}_{n+1}-(n+\alpha)\,B^{\alpha}_{n}-2xDB^{\alpha}_{n}+\gamma DB^{\alpha}_{n-1}=0.$$

In the same way, eliminating DB^{α}_{n+1} by taking a linear combination of relation (3.11) and (R2), we get

$$\text{(R3)}\quad 3(n+3)\,B^{\alpha}_{n+1}-2(x^2-3\gamma)\,DB^{\alpha}_{n}-(n+3\alpha)\,xB^{\alpha}_{n}+(\gamma x+9\delta)\,DB^{\alpha}_{n-1}=0,$$

and then differentiating (R3) and eliminating DB^{α}_{n+1}, we obtain

$$\text{(R4)}\qquad \begin{aligned}&-2(x^2-3\gamma)\,D^2B^{\alpha}_{n}+(n-3\alpha-2)\,xDB^{\alpha}_{n}+n(n+3\alpha)\,B^{\alpha}_{n}\\ &\qquad-n\gamma DB^{\alpha}_{n-1}+(\gamma x+9\delta)\,D^2B^{\alpha}_{n-1}=0.\end{aligned}$$

Using (R2) and (3.11), we eliminate DB^{α}_{n-1} and replace n by $n-1$; we obtain

$$\text{(R5)}\quad (\gamma x+9\delta)\,DB^{\alpha}_{n}-n\gamma B^{\alpha}_{n}-2(3\delta x+\gamma^2)\,DB^{\alpha}_{n-1}-3(n+3\alpha-1)\,\delta B^{\alpha}_{n-1}=0.$$

Differentiating (R5) and eliminating $D^2B^\alpha_{n-1}$ using (R4), we have

$$\text{(R6)}\quad S_3(x)\,D^2B^\alpha_n+[2(n-2-3\alpha)(3\delta x^2+\gamma^2 x)-(n-1)(\gamma x+9\delta)\gamma]\,DB^\alpha_n$$
$$+2n(n+3\alpha)(3\delta x+\gamma^2)\,B^\alpha_n-r_{1,n}(x)\,DB^\alpha_{n-1}=0.$$

Finally we differentiate (R6) and eliminate $D^2B^\alpha_{n-1}$ and then DB^α_{n-1} by combining it with relations (R4) and (R6) to obtain Eq. (5.15). ∎

THEOREM 5.2. *The zero of* $r_{1,n}(x)$ *is an apparent singularity of the differential equation* (5.15).

Proof. (We use the same notation as that given in [8].) Set $X=x-r^*$, where r^* is the zero of $r_{1,n}(x)$ $(r^*=(-2n\gamma^3+27(n+1+3\alpha)\,\delta^2)/3(3n+1+3\alpha)\,\gamma\delta)$. Then Eq. (5.15) can be written in the form

$$\left[S_3(r^*)+DS_3(r^*)\,X+\frac{D^2S_3(r^*)}{2}X^2+\frac{D^3S_3(r^*)}{6}X^3\right]X^3Y^{(3)}+\frac{1}{Dr_{1,n}}$$
$$+\left[b_{3,n}(r^*)+Db_{3,n}(r^*)\,X+\frac{D^2b_{3,n}(r^*)}{2}X^2+\frac{D^3b_{3,n}(r^*)}{6}X^3\right]X^2Y''$$
$$+\frac{1}{Dr_{1,n}}\left[c_{2,n}(r^*)\,X+Dc_{2,n}(r^*)\,X^2+\frac{D^2c_{2,n}(r^*)}{2}X^3\right]XY'$$
$$+\frac{1}{Dr_{1,n}}[d_{1,n}(r^*)\,X^2+Dd_{1,n}(r^*)\,X^3]\;Y=0.$$

Then

$$\left.\begin{aligned}
f_0(\rho)&=\rho(\rho-1)(\rho-2)\,S_3(r^*)+\rho(\rho-1)\,\frac{b_{3,n}(r^*)}{Dr_{1,n}},\\
f_1(\rho)&=\rho(\rho-1)(\rho-2)\,DS_3(r^*)+\rho(\rho-1)\,\frac{Db_{3,n}(r^*)}{Dr_{1,n}}+\rho\,\frac{c_{2,n}(r^*)}{Dr_{1,n}},\\
f_2(\rho)&=\rho(\rho-1)(\rho-2)\,\frac{D^2S_3(r^*)}{2}+\rho(\rho-1)\,\frac{D^2b_{3,n}(r^*)}{2Dr_{1,n}}\\
&\quad+\rho\,\frac{Dc_{2,n}(r^*)}{Dr_{1,n}}+\frac{d_{1,n}(x)}{Dr_{1,n}},\\
f_3(\rho)&=\rho(\rho-1)(\rho-2)\,\frac{D^3S_3(r^*)}{6}+\rho(\rho-1)\,\frac{D^2b_{3,n}(r^*)}{6Dr_{1,n}}\\
&\quad+\rho\,\frac{D^2c_{2,n}(r^*)}{2Dr_{1,n}}+\frac{Dd_{1,n}(x)}{Dr_{1,n}}.
\end{aligned}\right\}$$

Therefore, the indicial equation relative to $x = r^*$ is

$$f_0(\rho) = \rho(\rho-1)\left[(\rho-2)\, S_3(r^*) + \frac{b_{3,n}(r^*)}{Dr_{1,n}}\right] = \rho(\rho-1)(\rho-3)\, S_3(r^*) = 0,$$

and consequently, the exponents of Frobenius are

$$\rho_0 = 1,\ \rho_1 = 2, \qquad \text{and} \qquad \rho_2 = 3.$$

Since

$$\rho_1 - \rho_2 = 1 \qquad \text{and} \qquad \rho_0 - \rho_2 = 3,$$

the necessary and sufficient conditions are that

$$F_1(0) = 0; \qquad F_3(0) = 0; \qquad \left.\frac{\partial F_3}{\partial \rho}\right|_{\rho=0} = 0.$$

We have

$$F_1(0) = f_1(0) = 0,$$

and

$$F_3(\rho) = f_1(\rho)\, f_1(\rho+1)\, f_1(\rho+2) + f_0(\rho+1)\, f_0(\rho+2)\, f_3(\rho)$$
$$- f_0(\rho+1)\, f_1(\rho+2)\, f_2(\rho) - f_2(\rho+1)\, f_0(\rho)\, f_1(\rho).$$

Thus $F_3(\rho) = 0$ because $f_1(0) = f_0(1) = 0$, and

$$\left.\frac{\partial F_3}{\partial \rho}\right|_{\rho=0} = f_1'(0)[\, f_1(1)\, f_1(2) - f_2(1)\, f_0(2)],$$

because $f_1(0) = f_0(1) = f_0'(1) = 0$.

Since

$$f_1(1)\, f_1(2) - f_2(1)\, f_0(2) = 2\left\{\frac{c_{2,n}(r^*)}{[Dr_{1,n}]^2}\, [Db_{3,n}(r^*) + c_{2,n}(r^*)]\right.$$
$$\left. - \frac{b_{3,n}(r^*)}{[Dr_{1,n}]^2}\, [Dc_{2,n}(r^*) + d_{1,n}(x)]\right\} = 0,$$

we have

$$\left.\frac{\partial F_3}{\partial \rho}\right|_{\rho=0} = 0.$$

Remarks. (a) This result constitutes an extension of Hahn's theory [10].

(b) The solution of (5.15) is analytic at $x = r^*$ and can be written in the form

$$Y(x) = x^{\rho} \sum_{k \geqslant 0} g_k x^k, \tag{5.17}$$

where the coefficients g_k satisfy the following recurrence relations:

$$\left.\begin{aligned}
&g_0 f_0(\rho) = 0\\
&g_1 f_0(\rho+1) + g_0 f_1(\rho) = 0\\
&g_2 f_0(\rho+2) + g_1 f_1(\rho+1) + g_0 f_2(\rho) = 0\\
&g_m(\rho+m) + g_{m-1} f_1(\rho+m-1) + g_{m-2} f_2(\rho+m-2)\\
&\qquad + g_{m-3} f_3(\rho+m-3) = 0, \qquad m \geqslant 3.
\end{aligned}\right\} \tag{5.18}$$

6. STUDY OF THE PARTICULAR CASE WHEN $\gamma = 0$

THEOREM 6.1. *When $\gamma = 0$ (i.e., the sequence $\{B_n^{\alpha}\}_{n \geqslant 0}$ is 2-symmetric), the differential equation (5.15) becomes a differential equation of a hypergeometric type, where the solutions are hypergeometric functions ${}_3F_2$.*

Proof. It is straightforward to show that when $\gamma = 0$, Eq. (5.15) can be written as

$$\begin{aligned}
&(27\delta - 4x^3)\, Y^{(3)} - 6(2\alpha+3)\, x^2 Y'' + [3n(n+2\alpha+1) - (3\alpha+2)(3\alpha+5)]\, xY'\\
&\quad + n(n+3\alpha)(n+3\alpha+3)\, Y = 0.
\end{aligned} \tag{6.1}$$

By changing the variable $4x^3 = 27\delta X$ and putting $X(d/dX) = \theta$, Eq. (6.1) can be written in the form

$$\left[\theta\left(\theta - \frac{1}{3}\right)\left(\theta - \frac{2}{3}\right) - X\left(\theta - \frac{n}{3}\right)\left(\theta + \frac{n}{6} + \frac{\alpha}{2}\right)\left(\theta + \frac{n}{6} + \frac{\alpha+1}{2}\right)\right] Y = 0,$$

which is a hypergeometric differential equation, with solutions

$$Y_1(x) = \text{constant}\,{}_3F_2\left(-\frac{n}{3}, \frac{n}{6}+\frac{\alpha}{2}, \frac{n}{6}+\frac{\alpha+1}{2}; \frac{1}{3}, \frac{2}{3}; X\right) \tag{6.2}$$

$$Y_2(x) = \text{constant}\, X^{1/3}\,{}_3F_2\left(-\frac{n-1}{3}, \frac{n}{6}+\frac{\alpha}{2}+\frac{1}{3}, \frac{n}{6}+\frac{\alpha+1}{2}+\frac{1}{3}; \frac{2}{3}, \frac{4}{3}; X\right) \tag{6.3}$$

and

$$Y_3(x) = \text{constant}\, X^{2/3}\,{}_3F_2\left(-\frac{n-2}{3}, \frac{n}{6}+\frac{\alpha}{2}+\frac{2}{3}, \frac{n}{6}+\frac{\alpha+1}{2}+\frac{2}{3}; \frac{4}{3}, \frac{5}{3}; X\right). \quad \blacksquare \tag{6.4}$$

Remarks. (a) Note that

$$\left.\begin{aligned} Y_1(x) &= B^{\alpha}_{3n}(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{[\alpha]_{n+2k}\,\delta^{n-k}}{(3k)!\,(n-k)!}\, x^{3k}, \\ Y_2(x) &= B^{\alpha}_{3n+1}(x) = x \sum_{k=0}^{n} (-1)^{n-k} \frac{[\alpha]_{n+2k+1}\,\delta^{n-k}}{(3k+1)!\,(n-k)!}\, x^{3k}, \\ Y_3(x) &= B^{\alpha}_{3n+2}(x) = x^2 \sum_{k=0}^{n} (-1)^{n-k} \frac{[\alpha]_{n+2k+2}\,\delta^{n-k}}{(3k+2)!\,(n-k)!}\, x^{3k}, \end{aligned}\right\} \tag{6.5}$$

with

$$\left.\begin{aligned} &[\alpha]_n = \alpha(\alpha-1)\cdots(\alpha+1-n), \qquad n \geqslant 1; \\ &[\alpha]_0 = 1. \end{aligned}\right\}$$

This gives us an explicit form of the polynomials $B^{\alpha}_n(x)$, which may be written in the form

$$B^{\alpha}_n(x) = \sum_{k=0}^{[n/3]} (-1)^k \frac{[\alpha]_{n-2k}\,\delta^k}{(k)!\,(n-3k)!}\, x^{n-3k}. \tag{6.6}$$

(b) If we put

$$B^{\alpha}_{3n}(x) = D^{\alpha}_n(x^3),$$

$$B^{\alpha}_{3n+1}(x) = xE^{\alpha}_n(x^3), \qquad \text{and} \qquad B^{\alpha}_{3n+2}(x) = x^2F^{\alpha}_n(x^3), \tag{6.7}$$

it is easy to show that each of the sequences $\{D^{\alpha}_n\}_{n\geqslant 0}$, $\{E^{\alpha}_n\}_{n\geqslant 0}$, and $\{F^{\alpha}_n\}_{n\geqslant 0}$ also satisfies a recurrence relation of order 3.

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